

# The multicomponent 2D Toda hierarchy: dispersionless limit

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April 21, 2009

## Abstract

The factorization problem of the multi-component 2D Toda hierarchy is used to analyze the dispersionless limit of this hierarchy. A dispersive version of the Whitham hierarchy defined in terms of scalar Lax and Orlov–Schulman operators is introduced and the corresponding additional symmetries and string equations are discussed. Then, it is shown how KP and Toda pictures of the dispersionless Whitham hierarchy emerge in the dispersionless limit. Moreover, the additional symmetries and string equations for the dispersive Whitham hierarchy are studied in this limit.

## 1 Introduction

In [1] the theory of the multi-component Toda hierarchy [2] was analyzed from the point of view of a factorization problem

$$g = W^{-1} \bar{W} \quad (1)$$

in an infinite-dimensional group and a natural formulation of the additional symmetries and the string equations of the hierarchy was given. In the present work we use this formulation to study the dispersionless limit of the solutions of (1). As it is known in the theory of random matrix models [3]–[5], the study the large  $N$  limit can be performed in terms of the dispersionless limit of the string equations satisfied by the solution of the underlying integrable system. Notice that in recent years the formalism of string equations [6] for dispersionless integrable systems [7] has been much developed [8]. Our present work is motivated by the applications of multi-component integrable hierarchies [2, 9] to the study of the large  $N$  limit of the two-matrix model [10]–[13], as well as models of random matrices with external source and non-intersecting Brownian motions [14]–[19]. A common feature of these models is that they have an associated family of multiple orthogonal polynomials which is in turn characterized by a matrix Riemann–Hilbert (MRH) problem which is a basic ingredient to analyze the large  $N$  limit [19]–[22]. On the other hand, MRH problems also provide solutions of reductions of multi-component integrable hierarchies of KP or Toda type. These reductions correspond to solutions of factorization problems (1) constrained by certain types of string equations.

In our analysis we introduce matrix wave functions and scalar Lax and Orlov–Schulman operators [23] associated to the solutions of (1). We prove that the rows of the matrix wave functions satisfy auxiliary linear systems involving the scalar Lax operators, which constitute the dispersive versions of the genus zero dispersionless Whitham hierarchies [24]. In order to study the dispersionless limit, we assume the Takasaki–Takebe quasi-classical ansatz [25, 26] for the rows of the matrix wave functions. Thus, we prove that in the dispersionless limit the auxiliary linear systems reduce to systems of Hamilton–Jacobi equations that are shown to be equivalent to the dispersionless Whitham hierarchies. In particular, two natural pictures (KP and Toda types) of the dispersionless Whitham hierarchies emerge in our analysis. An important advantage of our approach is that it yields a natural method for characterizing string equations and additional symmetries in the dispersionless limit. In particular, we characterize the dispersive analogues of the soluble string equations discussed in [27].

The layout of the paper is as follows. In §1.1 we present a summary of the relevant parts of [1] needed in the subsequent analysis. Then, in §2 we discuss the dispersive Whitham hierarchies. We introduce a set of scalar

Lax and Orlov–Schulman operators, and vector wave functions to deduce the corresponding auxiliary linear systems, as well as additional symmetries and string equations of dispersive type. Finally, in §3 we discuss the aforementioned dispersionless limits. We find the Hamilton–Jacobi type equations, and then derive the KP and Toda pictures of the dispersionless Whitham hierarchy. We conclude the paper by considering the dispersionless counterparts of dispersive string equations.

## 1.1 Reminder

As in our previous work [1] we only consider formal series expansions in the Lie group theoretic set up without any assumption on their convergency. Let us remind some notations and results from [1]. Given Lie algebras  $\mathfrak{g}_1 \subset \mathfrak{g}_2$ , and  $X, Y \in \mathfrak{g}_2$  then  $X = Y + \mathfrak{g}_1$  means  $X - Y \in \mathfrak{g}_1$ . For any Lie groups  $G_1 \subset G_2$  and  $a, b \in G_2$  then  $a = G_2 \cdot b$  stands for  $a \cdot b^{-1} \in G_2$ .

Let  $M_N(\mathbb{C})$  denote the associative algebra of complex  $N \times N$  complex matrices we will consider the linear space of sequences  $f : \mathbb{Z} \rightarrow M_N(\mathbb{C})$ . The shift operator  $\Lambda$  acts on these sequences as  $(\Lambda f)(n) := f(n+1)$ . A sequence  $X : \mathbb{Z} \rightarrow M_N(\mathbb{C})$  acts by left multiplication in this space of sequences, and therefore we may consider operators of the type  $X\Lambda^j$ ,  $(X\Lambda^j)(f)(n) := X(n) \cdot f(n+j)$ .

Moreover, defining the product  $(X(n)\Lambda^i) \cdot (Y(n)\Lambda^j) := X(n)Y(n+i)\Lambda^{i+j}$  and extending it linearly we have that the set  $\mathfrak{g}$  of Laurent series in  $\Lambda$  is an associative algebra, which under the standard commutator is a Lie algebra.

This Lie algebra has the following important splitting

$$\mathfrak{g} = \mathfrak{g}_+ \dot{+} \mathfrak{g}_-, \quad (2)$$

where

$$\mathfrak{g}_+ = \left\{ \sum_{j \geq 0} X_j(n)\Lambda^j, \quad X_j(n) \in M_N(\mathbb{C}) \right\}, \quad \mathfrak{g}_- = \left\{ \sum_{j < 0} X_j(n)\Lambda^j, \quad X_j(n) \in M_N(\mathbb{C}) \right\},$$

are Lie subalgebras of  $\mathfrak{g}$  with trivial intersection.

The group of linear invertible elements in  $\mathfrak{g}$  will be denoted by  $G$  and has  $\mathfrak{g}$  as its Lie algebra, then the splitting (2) leads us to consider the following factorization of  $g \in G$

$$g = g_-^{-1} \cdot g_+, \quad g_{\pm} \in G_{\pm} \quad (3)$$

where  $G_{\pm}$  have  $\mathfrak{g}_{\pm}$  as their Lie algebras. Explicitly,  $G_+$  is the set of invertible linear operators of the form  $\sum_{j \geq 0} g_j(n)\Lambda^j$ ; while  $G_-$  is the set of invertible linear operators of the form  $1 + \sum_{j < 0} g_j(n)\Lambda^j$ .

Now we introduce two sets of indexes,  $\mathbb{S} = \{1, \dots, N\}$  and  $\bar{\mathbb{S}} = \{\bar{1}, \dots, \bar{N}\}$ , of the same cardinality  $N$ . In what follows we will use letters  $k, l$  and  $\bar{k}, \bar{l}$  to denote elements in  $\mathbb{S}$  and  $\bar{\mathbb{S}}$ , respectively. Furthermore, we will use letters  $a, b, c$  to denote elements in  $\mathcal{S} := \mathbb{S} \cup \bar{\mathbb{S}}$ .

We define the following operators  $W_0, \bar{W}_0 \in G$

$$W_0 := \sum_{k=1}^N E_{kk} \Lambda^{s_k} e^{\sum_{j=0}^{\infty} t_{jk} \Lambda^j}, \quad (4)$$

$$\bar{W}_0 := \sum_{k=1}^N E_{kk} \Lambda^{-s_{\bar{k}}} e^{\sum_{j=1}^{\infty} t_{j\bar{k}} \Lambda^{-j}} \quad (5)$$

where  $s_a \in \mathbb{Z}$ ,  $t_{ja} \in \mathbb{C}$  are deformation parameters, that in the sequel will play the role of discrete and continuous times, respectively. Given an element  $g \in G$  and a set of deformation parameters  $\mathbf{s} = (s_a)_{a \in \mathcal{S}}$ ,  $\mathbf{t} = (t_{ja})_{a \in \mathcal{S}, j \in \mathbb{N}}$  we will consider the factorization problem

$$S(\mathbf{s}, \mathbf{t}) \cdot W_0 \cdot g = \bar{S}(\mathbf{s}, \mathbf{t}) \cdot \bar{W}_0, \quad S \in G_- \text{ and } \bar{S} \in G_+, \quad (6)$$

and will confine ourselves to the *zero charge sector*  $|\mathbf{s}| := \sum_{a \in \mathcal{S}} s_a = 0$ . We define the dressing or Sato operators  $W, \bar{W}$  as follows

$$W := S \cdot W_0, \quad \bar{W} := \bar{S} \cdot \bar{W}_0, \quad (7)$$

so that the factorization problem in  $G$  reads

$$W \cdot g = \bar{W} \quad (8)$$

Observe that  $S, \bar{S}$  have expansions of the form

$$\begin{aligned} S &= \mathbb{I}_n + \varphi_1(n)\Lambda^{-1} + \varphi_2(n)\Lambda^{-2} + \cdots \in G_-, \\ \bar{S} &= \bar{\varphi}_0(n) + \bar{\varphi}_1(n)\Lambda + \bar{\varphi}_2(n)\Lambda^2 + \cdots \in G_+. \end{aligned} \quad (9)$$

The Lax operators  $L, \bar{L}, C_{kk}, \bar{C}_{kk} \in \mathfrak{g}$  are defined by

$$L := W \cdot \Lambda \cdot W^{-1}, \quad \bar{L} := \bar{W} \cdot \Lambda \cdot \bar{W}^{-1}, \quad (10)$$

$$C_{kk} := W \cdot E_{kk} \cdot W^{-1}, \quad \bar{C}_{kk} := \bar{W} \cdot E_{kk} \cdot \bar{W}^{-1} \quad (11)$$

and have the following expansions

$$\begin{aligned} L &= \Lambda + u_1(n) + u_2(n)\Lambda^{-1} + \cdots, & \bar{L}^{-1} &= \bar{u}_0(n)\Lambda^{-1} + \bar{u}_1(n) + \bar{u}_2(n)\Lambda + \cdots, \\ C_{kk} &= E_{kk} + C_{kk,1}(n)\Lambda^{-1} + C_{kk,2}(n)\Lambda^{-2} + \cdots, & \bar{C}_{kk} &= \bar{C}_{kk,0}(n) + \bar{C}_{kk,1}(n)\Lambda + \bar{C}_{kk,2}(n)\Lambda^2 + \cdots. \end{aligned} \quad (12)$$

Now we introduce some further notation

1.

$$\partial_{ja} := \frac{\partial}{\partial t_{ja}}, \quad \text{for } a = \mathcal{S} \text{ and } j = 1, 2, \dots$$

2. Given  $K = (a, b)$  the basic *charge preserving* shift operators  $T_K$  are defined as follows

$$(T_K f)(s_a, s_b) := f(s_a + 1, s_b - 1).$$

We define the Orlov–Schulman operators [23] for the multi-component 2D Toda hierarchy by

$$M := WnW^{-1}, \quad \bar{M} := \bar{W}n\bar{W}^{-1}. \quad (13)$$

One proves at once that

- The Orlov–Schulman operators satisfy the following commutation relations

$$[L, M] = L, \quad [L, C_{kk}] = 0, \quad [\bar{L}, \bar{M}] = \bar{L}, \quad [\bar{L}, \bar{C}_{kk}] = 0, \quad (14)$$

- The following expansions hold

$$\begin{aligned} M &= \mathcal{M} + \sum_{k=1}^N C_{kk}(s_k + \sum_{j=1}^{\infty} jt_{jk}L^j), & \mathcal{M} &= n + \mathfrak{g}_- \\ \bar{M} &= \bar{\mathcal{M}} - \sum_{k=1}^N \bar{C}_{kk}(s_{\bar{k}} + \sum_{j=1}^{\infty} jt_{j\bar{k}}\bar{L}^{-j}), & \bar{\mathcal{M}} &= n + \mathfrak{g}_+\Lambda. \end{aligned} \quad (15)$$

### 1.1.1 Additional symmetries

Suppose that the operator  $g$  in (8) depends on an additional parameter  $\epsilon \in \mathbb{C}$ . Then, the basic objects of the multi-component Toda hierarchy inherit a dependence on  $\epsilon$ . For convenience and for the time being we use the following equivalent factorization problem

$$W \cdot h = \bar{W} \cdot \bar{h},$$

with

$$g = h \cdot \bar{h}^{-1}. \quad (16)$$

Observe that

$$\begin{aligned}\partial_b W \cdot W^{-1} + W(\partial_b h \cdot h^{-1})W^{-1} &= \partial_b S \cdot S^{-1} + W(\partial_b h \cdot h^{-1})W^{-1} \\ &= \partial_b \bar{S} \cdot \bar{S}^{-1} + \bar{W}(\partial_b \bar{h} \cdot \bar{h}^{-1})\bar{W}^{-1} = \partial_b \bar{W} \cdot \bar{W}^{-1} + \bar{W}(\partial_b \bar{h} \cdot \bar{h}^{-1})\bar{W}^{-1}.\end{aligned}\quad (17)$$

Now, let us suppose that  $h$  and  $\bar{h}$  satisfy

$$\partial_b h \cdot h^{-1} = F^{(0)} = \sum_{l=1}^N F_l(n, \Lambda) E_{ll}, \quad \partial_b \bar{h} \cdot \bar{h}^{-1} = \bar{F}^{(0)} = \sum_{l=1}^N \bar{F}_l(n, \Lambda) E_{ll}, \quad (18)$$

then from (17) we get

$$\partial_b W \cdot W^{-1} = \partial_b S \cdot S^{-1} = -H_-, \quad \partial_b \bar{W} \cdot \bar{W}^{-1} = \partial_b \bar{S} \cdot \bar{S}^{-1} = H_+, \quad H_{\pm} \in \mathfrak{g}_{\pm}.$$

where

$$H := F - \bar{F}, \quad F := \sum_{l=1}^N F_l(M, L) C_{ll}, \quad \bar{F} := \sum_{l=1}^N \bar{F}_l(\bar{M}, \bar{L}) \bar{C}_{ll}. \quad (19)$$

Hence it follows that

**Proposition 1.** *Given a dependence on an additional parameter  $b$  according to (16), (18) and (19) then*

1. *The dressing operators  $W$  and  $\bar{W}$  satisfy*

$$\partial_b W = -H_- \cdot W, \quad \partial_b \bar{W} = H_+ \cdot \bar{W},$$

2. *The Lax and Orlov–Schulman operators satisfy*

$$\begin{aligned}\partial_b L &= [-H_-, L], & \partial_b M &= [-H_-, M], & \partial_b C_{kk} &= -[H_-, C_{kk}], \\ \partial_b \bar{L} &= [H_+, \bar{L}], & \partial_b \bar{M} &= [H_+, \bar{M}], & \partial_b \bar{C}_{kk} &= [H_+, \bar{C}_{kk}].\end{aligned}\quad (20)$$

A key observation is

**Proposition 2.** *Given operators  $R, \bar{R} \in \mathfrak{g}$  satisfying  $R \cdot g = \bar{R}$  and such that*

$$\begin{aligned}RW_0^{-1} &\in \mathfrak{g}_-, \\ \bar{R}\bar{W}_0^{-1} &\in \mathfrak{g}_+.\end{aligned}\quad (21)$$

Then  $R = \bar{R} = 0$

## 1.2 Wave functions

The wave functions of the multi-component 2D Toda hierarchy are defined by

$$\psi = W \cdot \chi, \quad \bar{\psi} = \bar{W} \cdot \chi. \quad (22)$$

where

$$\chi(z) := \{z^n \mathbb{I}_N\}_{n \in \mathbb{Z}},$$

Note that  $\Lambda \chi = z \chi$ . The following asymptotic expansions are a consequence of (9)

$$\begin{aligned}\psi &= z^n (\mathbb{I}_N + \varphi_1(n) z^{-1} + \cdots) \psi_0(z), & \psi_0 &:= \sum_{k=1}^N E_{kk} z^{s_k} e^{\sum_{j=1}^{\infty} t_{jk} z^j}, & z &\rightarrow \infty, \\ \bar{\psi} &= z^n (\bar{\varphi}_0(n) + \bar{\varphi}_1(n) z + \cdots) \bar{\psi}_0(z), & \bar{\psi}_0 &:= \sum_{k=1}^N E_{kk} z^{-s_k} e^{\sum_{j=1}^{\infty} t_{j\bar{k}} z^{-j}}, & z &\rightarrow 0.\end{aligned}\quad (23)$$

**Proposition 3.** 1. Given operators of the form

$$F := \sum_{k=1}^N F_k C_{kk}, \quad F_k := \sum_{i \geq 0, j \in \mathbb{Z}} F_{kij} M^i L^j, \quad \bar{F} := \sum_{k=1}^N F_{\bar{k}} \bar{C}_{kk}, \quad F_{\bar{k}} := \sum_{i \geq 0, j \in \mathbb{Z}} F_{\bar{k}ij} \bar{M}^i \bar{L}^j,$$

with complex-valued scalar coefficients, we have

$$F(\psi) = (\psi) \sum_{k=1}^N \overleftarrow{F_k \left( z \frac{d}{dz}, z \right)} E_{kk}, \quad \bar{F}(\bar{\psi}) = (\bar{\psi}) \sum_{k=1}^N \overleftarrow{F_{\bar{k}} \left( z \frac{d}{dz}, z \right)} E_{kk}. \quad (24)$$

where

$$(\psi) \overleftarrow{F_k \left( z \frac{d}{dz}, z \right)} := \sum_{i \geq 0, j \in \mathbb{Z}} F_{kij} z^j \left( z \frac{d}{dz} \right)^i (\psi), \quad (\bar{\psi}) \overleftarrow{F_{\bar{k}} \left( z \frac{d}{dz}, z \right)} := \sum_{i \geq 0, j \in \mathbb{Z}} \bar{F}_{\bar{k}ij} z^j \left( z \frac{d}{dz} \right)^i (\bar{\psi})$$

2. Given operators

$$\begin{aligned} P &:= \sum_{k=1}^N P_k C_{kk}, & P_k &:= \sum_{i \geq 0, j \in \mathbb{Z}} P_{kij} M^i L^j, & Q &:= \sum_{k=1}^N Q_k C_{kk}, & Q_k &:= \sum_{i \geq 0, j \in \mathbb{Z}} Q_{kij} M^i L^j, \\ \bar{P} &:= \sum_{k=1}^N P_{\bar{k}} \bar{C}_{kk}, & P_{\bar{k}} &:= \sum_{i \geq 0, j \in \mathbb{Z}} P_{\bar{k}ij} \bar{M}^i \bar{L}^j, & \bar{Q} &:= \sum_{k=1}^N Q_{\bar{k}} \bar{C}_{kk}, & Q_{\bar{k}} &:= \sum_{i \geq 0, j \in \mathbb{Z}} \bar{Q}_{\bar{k}ij} \bar{M}^i \bar{L}^j, \end{aligned}$$

with complex-valued scalar coefficients, we have

$$PQ(\psi) = \sum_{k=1}^N \left( (\psi) \overleftarrow{P_k \left( z \frac{d}{dz}, z \right)} \right) \overleftarrow{Q_k \left( z \frac{d}{dz}, z \right)}, \quad \bar{P}\bar{Q}(\bar{\psi}) = \sum_{k=1}^N \left( (\bar{\psi}) \overleftarrow{P_{\bar{k}} \left( z \frac{d}{dz}, z \right)} \right) \overleftarrow{Q_{\bar{k}} \left( z \frac{d}{dz}, z \right)}.$$

*Proof.* 1. It is easy to find from (10) and (15) that

$$\begin{aligned} M^i L^j C_{kk}(\psi) &= W n^i \Lambda^j E_{kk}(\chi) = W \left( \{n^i z^{n+j} \mathbb{I}_N\}_{n \in \mathbb{Z}} \right) E_{kk}, \\ \bar{M}^i \bar{L}^j \bar{C}_{kk}(\bar{\psi}) &= \bar{W} n^i \Lambda^j E_{kk}(\chi) = \bar{W} \left( \{n^i z^{n+j} \mathbb{I}_N\}_{n \in \mathbb{Z}} \right) E_{kk}. \end{aligned}$$

Now observe that the action of  $X = \sum_{j' \in \mathbb{Z}} X_{j'} \Lambda^{j'}$  on  $\{n^i z^{n+j}\}_{n \in \mathbb{Z}}$  is

$$X \left( \{n^i z^{n+j}\}_{n \in \mathbb{Z}} \right) = \left\{ \sum_{j' \in \mathbb{Z}} X_{j'}(n) (n+j')^i z^{n+j+j'} \right\}_{n \in \mathbb{Z}}$$

or equivalently

$$z^j \left( z \frac{d}{dz} \right)^i (X \cdot \chi) = \left\{ z^j \left( z \frac{d}{dz} \right)^i \left( \sum_{j' \in \mathbb{Z}} X_{j'}(n) z^{n+j'} \right) \right\}_{n \in \mathbb{Z}}.$$

Thus, the formulae

$$M^i L^j C_{kk}(\psi) = z^j \left( z \frac{d}{dz} \right)^i (\psi) E_{kk}, \quad \bar{M}^i \bar{L}^j \bar{C}_{kk}(\bar{\psi}) = z^j \left( z \frac{d}{dz} \right)^i (\bar{\psi}) E_{kk}, \quad (25)$$

hold.

2. It is a consequence of the identities

$$\begin{aligned} M^{i_1} L^{j_1} M^{i_2} L^{j_2} &= M^{i_1} (M + j_1)^{i_2} L^{j_1+j_2}, \\ \bar{M}^{i_1} \bar{L}^{j_1} \bar{M}^{i_2} \bar{L}^{j_2} &= \bar{M}^{i_1} (\bar{M} + j_1)^{i_2} \bar{L}^{j_1+j_2}, \\ z^{j_2} \left( z \frac{d}{dz} \right)^{i_2} z^{j_1} \left( z \frac{d}{dz} \right)^{i_1} &= z^{j_1+j_2} \left( z \frac{d}{dz} + j_1 \right)^{i_2} \left( z \frac{d}{dz} \right)^{i_1}, \end{aligned}$$

for any  $i_1, i_2 \geq 0$  and  $j_1, j_2 \in \mathbb{Z}$ . Therefore,

$$\left( (\psi) \overleftarrow{\left( z \frac{d}{dz} \right)^{i_1} z^{j_1}} \right) \overleftarrow{\left( z \frac{d}{dz} \right)^{i_2} z^{j_2}} = (\psi) \overleftarrow{\left( z \frac{d}{dz} \right)^{i_1} \left( z \frac{d}{dz} + j_1 \right)^{i_2} z^{j_1+j_2}} = M^{i_1} L^{j_1} M^{i_2} L^{j_2}(\psi).$$

□

## 2 The dispersive Whitham hierarchies

As we will see certain families of equations of the multi-component 2D Toda hierarchy, associated with any given row of the dressing operators, become the Whitham hierarchies under appropriate dispersionless limits. Consequently, these families will be referred to as the dispersive Whitham hierarchies.

For simplicity and without loss of generality, we will work with the first row of the dressing operators. It will be useful to introduce the following shift operators

$$\tau_a := \begin{cases} T_{(1,a_0)}, & a = 1, \\ T_{(a,1)}, & a \neq 1 \end{cases} \quad \bar{\tau}_a := \begin{cases} T_{(\bar{1},a_0)}, & a = \bar{1}, \\ T_{(a,\bar{1})}, & a \neq \bar{1}, \end{cases} \quad (26)$$

where for the cases  $a = 1$  and  $a = \bar{1}$ , the index  $a_0$  stands for any fixed elements in  $\mathcal{S} - \{1\}$  and  $\bar{\mathcal{S}} - \{\bar{1}\}$ , respectively. These two types of shift operators, that we refer as bared and unbared, lead to two algebras of shift operators, and also to two different families of Hamilton–Jacobi equations, see (47) and (48). We also define the scalar dressing operators

$$\mathcal{K}_a := \begin{cases} 1 + \varphi_{1,11}\tau_1^{-1} + \varphi_{2,11}\tau_l^{-2} + \cdots, & a = 1 \\ \varphi_{1,1k} + \varphi_{2,1k}\tau_k^{-1} + \cdots, & a = k \neq 1, \\ \bar{\varphi}_{0,1k} + \bar{\varphi}_{1,1k}\tau_k^{-1} + \cdots, & a = \bar{k}, \end{cases} \quad (27)$$

$$\bar{\mathcal{K}}_a := \begin{cases} 1 + \varphi_{1,11}\bar{\tau}_1^{-1} + \varphi_{2,11}\bar{\tau}_1^{-2} + \cdots, & a = 1, \\ \varphi_{1,lk} + \varphi_{2,lk}\bar{\tau}_k^{-1} + \cdots, & a = k \neq 1 \\ \bar{\varphi}_{0,1k} + \bar{\varphi}_{1,1k}\bar{\tau}_k^{-1} + \cdots, & a = \bar{k}, \end{cases} \quad (28)$$

where  $\varphi_i, \bar{\varphi}_i$  are the matrix coefficients of (9).

Thus, we may now introduce the associated scalar Lax operators

$$\begin{aligned} \mathcal{L}_a &:= \mathcal{K}_a \circ \tau_a \circ \mathcal{K}_a^{-1} = \mathcal{W}_a \circ \tau_a \circ \mathcal{W}_a^{-1} = \begin{cases} \tau_1 + L_{1,0} + L_{1,-1}\tau_1^{-1} + \cdots, & a = 1, \\ L_{a,1}\tau_a + L_{a,0} + L_{a,-1}\tau_a^{-1} + \cdots, & a \neq 1 \end{cases} \\ \bar{\mathcal{L}}_a &:= \bar{\mathcal{K}}_a \circ \bar{\tau}_a \circ \bar{\mathcal{K}}_a^{-1} = \bar{\mathcal{W}}_a \circ \bar{\tau}_a \circ \bar{\mathcal{W}}_a^{-1} = \begin{cases} \bar{\tau}_1 + \bar{L}_{1,0} + \bar{L}_{1,-1}\bar{\tau}_1^{-1} + \cdots, & a = 1, \\ \bar{L}_{a,1}\bar{\tau}_a + \bar{L}_{a,0} + \bar{L}_{a,-1}\bar{\tau}_a^{-1} + \cdots, & a \neq 1, \end{cases} \end{aligned} \quad (29)$$

where

$$\mathcal{W}_a := \mathcal{K}_a \circ \mathcal{W}_{0,a}, \quad \mathcal{W}_{0,a} := \exp(\mathcal{T}_a), \quad \mathcal{T}_a := \sum_{j=1}^{\infty} t_{ja}\tau_a^j, \quad (30)$$

$$\bar{\mathcal{W}}_a := \bar{\mathcal{K}}_a \circ \bar{\mathcal{W}}_{0,a}, \quad \bar{\mathcal{W}}_{0,a} := \exp(\bar{\mathcal{T}}_a), \quad \bar{\mathcal{T}}_a := \sum_{j=1}^{\infty} t_{ja}\bar{\tau}_a^j. \quad (31)$$

Similarly, we define the corresponding scalar Orlov–Schulman operators by

$$\mathcal{M}_a := n - \nu_a + \text{sg}(a)\mathcal{W}_a \circ s_a \circ \mathcal{W}_a^{-1}, \quad \bar{\mathcal{M}}_a := n - \nu_a + \text{sg}(a)\bar{\mathcal{W}}_a \circ s_a \circ \bar{\mathcal{W}}_a^{-1}, \quad (32)$$

where

$$\text{sg}(a) := \begin{cases} 1, & a \in \mathbb{S}, \\ -1, & a \in \bar{\mathbb{S}}, \end{cases} \quad \nu_a := \begin{cases} 1, & a \in \mathbb{S} - \{1\}, \\ 0, & a \notin \mathbb{S} - \{1\}. \end{cases}$$

From the identities

$$[\tau_a, \text{sg}(a)s_a] = \text{sg}(a)\tau_a, \quad [\bar{\tau}_a, \text{sg}(a)s_a] = \text{sg}(a)\bar{\tau}_a,$$

it follows that

$$[\mathcal{L}_a, \mathcal{M}_a] = \text{sg}(a)\mathcal{L}_a, \quad [\bar{\mathcal{L}}_a, \bar{\mathcal{M}}_a] = \text{sg}(a)\bar{\mathcal{L}}_a,$$

**Proposition 4.** *The Orlov–Schulman operators satisfy*

$$\mathcal{M}_a = n - \nu_a + \text{sg}(a) \left( s_a + \sum_{j=1}^{\infty} j t_{ja} \mathcal{L}_a^j + \sum_{i=1}^{\infty} m_{ai} \mathcal{T}_a^{-i} \right), \quad (33)$$

$$\bar{\mathcal{M}}_a := n - \nu_a + \text{sg}(a) \left( s_a + \sum_{j=1}^{\infty} j t_{ja} \bar{\mathcal{L}}_a^j + \sum_{i=1}^{\infty} \bar{m}_{ai} \bar{\mathcal{T}}_a^{-i} \right). \quad (34)$$

*Proof.* These formulae follow from

$$\mathcal{W}_{0,a} s_a \mathcal{W}_{0,a}^{-1} = s_a + [\mathcal{T}_a, s_a] = s_a + \sum_{j=1}^{\infty} j \mathcal{T}_a^j, \quad \bar{\mathcal{W}}_{0,a} s_a \bar{\mathcal{W}}_{0,a}^{-1} = s_a + [\bar{\mathcal{T}}_a, s_a] = s_a + \sum_{j=1}^{\infty} j \bar{\mathcal{T}}_a^j,$$

and the fact that there are expansions of the form

$$\mathcal{K}_a s_a \mathcal{K}_a^{-1} = s_a + \sum_{i=1}^{\infty} m_{ai} \mathcal{T}_a^{-i}, \quad \bar{\mathcal{K}}_a s_a \bar{\mathcal{K}}_a^{-1} = s_a + \sum_{i=1}^{\infty} \bar{m}_{ai} \bar{\mathcal{T}}_a^{-i}.$$

□

We further introduce the vector wave functions

$$\Psi_a := \begin{cases} \psi_{1k}, & a = k, \\ \bar{\psi}_{1k}, & a = \bar{k}, \end{cases} \quad (35)$$

**Proposition 5.** *We have the identities*

$$[F_a(\mathcal{M}_a, \mathcal{L}_a)](\Psi_a) = [F_a(\bar{\mathcal{M}}_a, \bar{\mathcal{L}}_a)](\Psi_a) = (\Psi_a) \overleftarrow{F_a \left( z \frac{d}{dz}, z^{\text{sg } a} \right)} = \begin{cases} E_{11} F_k(M, L) C_{kk}(\psi), & a = k, \\ E_{11} F_{\bar{k}}(\bar{M}, \bar{L}^{-1}) \bar{C}_{kk}(\bar{\psi}), & a = \bar{k}. \end{cases} \quad (36)$$

*Proof.* From the definitions (4),(5) and (7)

$$\begin{aligned} W_{1k} n^i \Lambda^j &= S_{1k} W_{0,kk} n^i \Lambda^j = S_{1k} (W_{0,kk} n W_{0,kk}^{-1})^i \Lambda^j W_{0,kk} = S_{1k} (n + s_k + \sum_{j'=1}^{\infty} j' t_{j'k} \Lambda^{j'})^i \Lambda^j W_{0,kk}, \\ \bar{W}_{1k} n^i \Lambda^j &= \bar{S}_{1k} \bar{W}_{0,kk} n^i \Lambda^j = \bar{S}_{1k} (\bar{W}_{0,kk} n \bar{W}_{0,kk}^{-1})^i \Lambda^j \bar{W}_{0,kk} = \bar{S}_{1k} (n - s_{\bar{k}} - \sum_{j'=1}^{\infty} j' t_{j'\bar{k}} \Lambda^{-j'})^i \Lambda^j \bar{W}_{0,kk}, \end{aligned}$$

Now, observe that

$$\begin{aligned} \Lambda^{-1}(n + s_k) W_{0,kk} &= \mathcal{T}_k^{-1}((n + s_k) W_{0,kk}) = \bar{\mathcal{T}}_k^{-1}((n + s_k) W_{0,kk}), \\ \Lambda(n - s_{\bar{k}}) \bar{W}_{0,kk} &= \mathcal{T}_{\bar{k}}^{-1}((n - s_{\bar{k}}) \bar{W}_{0,kk}) = \bar{\mathcal{T}}_{\bar{k}}^{-1}((n - s_{\bar{k}}) \bar{W}_{0,kk}), \end{aligned}$$

together with Proposition 3 imply the result. □

## 2.1 Auxiliary linear systems

Our next analysis uses the following complex algebras

$$\mathfrak{t}_a := \left\{ \sum_{j \in \mathbb{Z}} c_j \mathcal{T}_a^j \right\}, \quad \bar{\mathfrak{t}}_a := \left\{ \sum_{j \in \mathbb{Z}} c_j \bar{\mathcal{T}}_a^j \right\}, \quad (37)$$

and their subalgebras

$$\begin{cases} \mathbf{t}_{a,+} = \mathbf{t}_{a,>} := \left\{ \sum_{j>0} c_j T_a^j \right\} & \mathbf{t}_{a,\leq} := \left\{ \sum_{j\leq 0} c_j T_a^j \right\}, & a \neq 1 \\ \mathbf{t}_{1,+} = \mathbf{t}_{1,\geq} := \left\{ \sum_{j\geq 0} c_j T_1^j \right\} & \mathbf{t}_{1,<} := \left\{ \sum_{j<0} c_j T_1^j \right\} \end{cases} \quad (38)$$

$$\begin{cases} \bar{\mathbf{t}}_{a,+} = \bar{\mathbf{t}}_{a,>} := \left\{ \sum_{j>0} c_j (\bar{T}_a^j - 1) \right\}, & \bar{\mathbf{t}}_{a,<} := \left\{ \sum_{j<0} c_j (\bar{T}_a^j - 1) \right\}, & a \neq 1, \bar{1} \\ \bar{\mathbf{t}}_{1,+} = \bar{\mathbf{t}}_{1,\geq} := \left\{ \sum_{j\geq 0} c_j \bar{T}_1^j \right\}, & \bar{\mathbf{t}}_{1,<} := \left\{ \sum_{j<0} c_j \bar{T}_1^j \right\}, \\ \bar{\mathbf{t}}_{\bar{1},+} = \bar{\mathbf{t}}_{\bar{1},>} := \left\{ \sum_{j>0} c_j (\bar{T}_{\bar{1}}^j - 1) \right\} & \bar{\mathbf{t}}_{\bar{1},<} := \left\{ \sum_{j<0} c_j (\bar{T}_{\bar{1}}^j - 1) \right\}, & a' \neq 1, \\ \bar{\mathbf{t}}_{\bar{1},+} = \bar{\mathbf{t}}_{\bar{1},\geq} := \left\{ \sum_{j\geq 0} c_j \bar{T}_{\bar{1}}^j \right\} & \bar{\mathbf{t}}_{\bar{1},\leq} := \left\{ \sum_{j\leq 0} c_j \bar{T}_{\bar{1}}^j \right\}, & a' = 1. \end{cases} \quad (39)$$

We will denote by  $(T_{a,+}, T_{a,<}, T_{a,>}, T_{a,\leq}, T_{a,\geq})$  the projections of an operator  $T_a$  induced by the corresponding splittings.

The following important result links the operators  $(\mathcal{M}_k, \mathcal{L}_k)$  with the operators  $(M, L)$ . Here the splittings for each shift algebra  $\mathbf{t}_a$  or  $\bar{\mathbf{t}}_a$  are those indicated by (38) and (39),

**Proposition 6.** *The following relations hold*

$$\begin{cases} F(\mathcal{M}_k, \mathcal{L}_k)_+(E_{11}W) = F(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)_+(E_{11}W) = E_{11}(F(M, L)C_{kk})_+W, \\ F(\mathcal{M}_k, \mathcal{L}_k)_+(E_{11}\bar{W}) = F(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)_+(E_{11}\bar{W}) = E_{11}(F(M, L)C_{kk})_+\bar{W}, \end{cases} \quad (40)$$

$$\begin{cases} F(\mathcal{M}_{\bar{k}}, \mathcal{L}_{\bar{k}})_+(E_{11}W) = F(\bar{\mathcal{M}}_{\bar{k}}, \bar{\mathcal{L}}_{\bar{k}})_+(E_{11}W) = E_{11}(F(\bar{M}, \bar{L}^{-1})\bar{C}_{kk})_-W, \\ F(\mathcal{M}_{\bar{k}}, \mathcal{L}_{\bar{k}})_+(E_{11}\bar{W}) = F(\bar{\mathcal{M}}_{\bar{k}}, \bar{\mathcal{L}}_{\bar{k}})_+(E_{11}\bar{W}) = E_{11}(F(\bar{M}, \bar{L}^{-1})\bar{C}_{kk})_-\bar{W}. \end{cases} \quad (41)$$

*Proof.* See Appendix B. □

If we set  $F(x, y) = y^j$  in Proposition 6 and recall that

$$\partial_{ja}W = B_{ja}W, \quad \partial_{ja}\bar{W} = B_{ja}\bar{W},$$

with  $B_{jk} = (C_{kk}L^j)_+$ ,  $B_{j\bar{k}} = (\bar{C}_{kk}\bar{L}^{-j})_-$  [1] we deduce

**Theorem 1.** *The following scalar linear systems hold*

$$\partial_{ja}(E_{11}W) = (\mathcal{L}_a^j)_+(E_{11}W) = (\bar{\mathcal{L}}_a^j)_+(E_{11}W), \quad \partial_{ja}(E_{11}\bar{W}) = (\mathcal{L}_a^j)_+(E_{11}\bar{W}) = (\bar{\mathcal{L}}_a^j)_+(E_{11}\bar{W}) \quad (42)$$

The linear system (42) determines a set of commuting flows for  $(W, \bar{W})$  which, as we will show in the next Section, leads to the Whitham hierarchy in the dispersionless limit. For that reason this system will be referred to as the dispersive Whitham hierarchy of flows.

## 2.2 Additional symmetries and string equations

Using Proposition 1 we deduce the following results on the additional symmetries

**Proposition 7.** *Given an additional symmetry*

$$\begin{aligned} \partial_s E_{11}W &= -E_{11} \left( \sum_{k=1}^N (F_k(M, L)C_{kk} - F_{\bar{k}}(\bar{M}, \bar{L}^{-1})\bar{C}_{kk}) \right)_- \cdot W, \\ \partial_s E_{11}\bar{W} &= E_{11} \left( \sum_{k=1}^N (F_k(M, L)C_{kk} - F_{\bar{k}}(\bar{M}, \bar{L}^{-1})\bar{C}_{kk}) \right)_+ \cdot \bar{W}, \end{aligned} \quad (43)$$



then we have

$$\begin{aligned}\partial_b(\Psi_a) &= -F_a(\mathcal{M}_a, \mathcal{L}_a)(\Psi_a) + \left( \sum_{a' \in \mathcal{S}} F_{a'}(\mathcal{M}_{a'}, \mathcal{L}_{a'})_+ \right)(\Psi_a) \\ &= -F_a(\bar{\mathcal{M}}_a, \bar{\mathcal{L}}_a)(\Psi_a) + \left( \sum_{a' \in \mathcal{S}} F_{a'}(\bar{\mathcal{M}}_{a'}, \bar{\mathcal{L}}_{a'})_+ \right)(\Psi_a).\end{aligned}$$

*Proof.* From (43) we get

$$\begin{aligned}\partial_b E_{11} W &= - \sum_{k=1}^N E_{11} F_k(M, L) C_{kk} \cdot W + E_{11} \left[ \left( \sum_{k=1}^N F_k(M, L) C_{kk} \right)_+ + \left( \sum_{k=1}^N \bar{F}_k(\bar{M}, \bar{L}^{-1}) \bar{C}_{kk} \right)_- \right] \cdot W, \\ \partial_b E_{11} \bar{W} &= - \sum_{k=1}^N E_{11} \bar{F}_k(\bar{M}, \bar{L}^{-1}) \bar{C}_{kk} \cdot \bar{W} + E_{11} \left[ \left( \sum_{k=1}^N F_k(M, L) C_{kk} \right)_+ + \left( \sum_{k=1}^N \bar{F}_k(\bar{M}, \bar{L}^{-1}) \bar{C}_{kk} \right)_- \right] \cdot \bar{W}.\end{aligned}$$

Now, from Propositions 5 and 6 we conclude that

$$\begin{aligned}\partial_b(E_{11} W) &= - \sum_{k=1}^N F_k(\mathcal{M}_k, \mathcal{L}_k)(W_{1k}) E_{1k} + \left( \sum_{k=1}^N (F_k(\mathcal{M}_k, \mathcal{L}_k)_+ + F_{\bar{k}}(\mathcal{M}_{\bar{k}}, \mathcal{L}_{\bar{k}})_+) \right)(E_{11} W) \\ &= - \sum_{k=1}^N F_k(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)(W_{1k}) E_{1k} + \left( \sum_{k=1}^N (F_k(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)_+ + F_{\bar{k}}(\bar{\mathcal{M}}_{\bar{k}}, \bar{\mathcal{L}}_{\bar{k}})_+) \right)(E_{11} W), \\ \partial_b(E_{11} \bar{W}) &= - \sum_{k=1}^N \bar{F}_k(\mathcal{M}_k, \mathcal{L}_k)(\bar{W}_{1k}) E_{1k} + \left( \sum_{k=1}^N (F_k(\mathcal{M}_k, \mathcal{L}_k)_+ + F_{\bar{k}}(\mathcal{M}_{\bar{k}}, \mathcal{L}_{\bar{k}})_+) \right)(E_{11} \bar{W}) \\ &= - \sum_{k=1}^N \bar{F}_k(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)(\bar{W}_{1k}) E_{1k} + \left( \sum_{k=1}^N (F_k(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)_+ + F_{\bar{k}}(\bar{\mathcal{M}}_{\bar{k}}, \bar{\mathcal{L}}_{\bar{k}})_+) \right)(E_{11} \bar{W}).\end{aligned}$$

and the result follows.  $\square$

As a consequence we have

**Proposition 8.** *If the string equation*

$$E_{11} \sum_{k=1}^N F_k(M, L) C_{kk} = E_{11} \sum_{k=1}^N \bar{F}_k(\bar{M}, \bar{L}^{-1}) \bar{C}_{kk} \quad (44)$$

*is satisfied, then*

$$F_a(\mathcal{M}_a, \mathcal{L}_a)(\Psi_a) = \left( \sum_{a' \in \mathcal{S}} F_{a'}(\mathcal{M}_{a'}, \mathcal{L}_{a'})_+ \right)(\Psi_a), \quad F_a(\bar{\mathcal{M}}_a, \bar{\mathcal{L}}_a)(\Psi_a) = \left( \sum_{a' \in \mathcal{S}} F_{a'}(\bar{\mathcal{M}}_{a'}, \bar{\mathcal{L}}_{a'})_+ \right)(\Psi_a),$$

for all  $a \in \mathcal{S}$ .

*Proof.* The string equations (44) imply the invariance conditions

$$\partial_b E_{11} W = \partial_b E_{11} \bar{W} = 0. \quad (45)$$

Now, recalling Proposition 7 we get the desired result.  $\square$

### 3 The dispersionless limit

We consider here the dispersionless limit of the multi-component 2D Toda hierarchy. For that aim we use the vector wave functions (35) at a given fixed value  $n_0$  of the discrete variable  $n$ . Thus, from Theorem 1 the following auxiliary linear system follows

$$\partial_{j,a}(\Psi_b) = (\mathcal{L}_a^j)_+(\Psi_b) = (\bar{\mathcal{L}}_a^j)_+(\Psi_b) \quad a \in \mathcal{S}, j = 1, 2, \dots \quad (46)$$

Let us now introduce slow variables by

$$t_{\text{sl},ja} = \epsilon t_{ja}, \quad s_{\text{sl},a} = \epsilon s_a,$$

where  $\epsilon$  is a small real parameter and  $s_{\text{sl},a}$  are assumed to be continuous variables. For the sake of simplicity, we will henceforth denote by  $(t_{ja}, s_a)$  these slow variables. Moreover, we assume that the wave functions have the quasi-classical form

$$\Psi_a = \exp\left(\frac{\mathcal{S}_a}{\epsilon}\right), \quad \mathcal{S}_a = \mathcal{S}_{a,0} + \epsilon \mathcal{S}_{a,1} + \dots$$

with

$$\mathcal{S}_a = \mathcal{T}_a + \begin{cases} \epsilon \varphi_{1,11} z^{-1} + O(z^{-2}) & a = 1, \\ \epsilon \log \varphi_{1,k1} + O(z^{-1}) & a = k \neq 1 \\ \epsilon \log \bar{\varphi}_{0,k\bar{1}} + O(z) & a = \bar{k}. \end{cases} \quad \mathcal{T}_a := \begin{cases} (\epsilon n_0 + s_1) \log z + \sum_{j=1}^{\infty} t_{j1} z^j, & a = 1, \\ (\epsilon n_0 + s_k - \epsilon) \log z + \sum_{j=1}^{\infty} t_{jk} z^j, & a = k \neq 1, \\ (\epsilon n_0 - s_{\bar{k}}) \log z + \sum_{j=1}^{\infty} t_{j\bar{k}} z^{-j}, & a = \bar{k}. \end{cases}$$

From these expressions we deduce that as  $\epsilon \rightarrow 0$

$$\begin{aligned} \varphi_{1,11} &= O(\epsilon^{-1}), \\ \log \varphi_{1,k1} &= O(\epsilon^{-1}), & k \neq 1, \\ \log \bar{\varphi}_{0,k\bar{1}} &= O(\epsilon^{-1}). \end{aligned}$$

As a consequence the coefficients in the operators  $\mathcal{L}_a, \bar{\mathcal{L}}_a$  are Taylor series in  $\epsilon$  while those of the Orlov–Schulman operators  $\mathcal{M}_a, \bar{\mathcal{M}}_a$  have at most a simple pole in  $\epsilon = 0$ .

We introduce some new variables

$$\begin{aligned} \sigma_a &:= s_a, & a \neq 1, & \quad \bar{\sigma}_a := s_a, & a \neq \bar{1}, \\ \sigma_1 &:= \sum_{a \in \mathcal{S}} s_a, & & \quad \bar{\sigma}_1 := \sum_{a \in \mathcal{S}} s_a, \end{aligned}$$

Observe that

$$\frac{\partial}{\partial \sigma_a} = \frac{\partial}{\partial s_a} - \frac{\partial}{\partial s_1}, \quad a \neq 1, \quad \frac{\partial}{\partial \bar{\sigma}_a} = \frac{\partial}{\partial s_a} - \frac{\partial}{\partial s_{\bar{1}}}, \quad a \neq \bar{1}.$$

The zero charge condition implies that  $\sigma_1 = \sigma_{\bar{1}} = 0$ . Then, we define

$$\partial_a := \begin{cases} \frac{\partial}{\partial \sigma_a}, & a \neq 1, \\ -\frac{\partial}{\partial \sigma_{a_0}}, & a = 1, \end{cases}, \quad \bar{\partial}_a := \begin{cases} \frac{\partial}{\partial \bar{\sigma}_a}, & a \neq \bar{1}, \\ -\frac{\partial}{\partial \bar{\sigma}_{a_0}}, & a = \bar{1}. \end{cases}$$

Notice that

**Proposition 9.** *In the limit  $\epsilon \rightarrow 0$  we have that*

$$\begin{aligned} \tau_a^j(\exp(\mathcal{S}_b/\epsilon)) &= \exp(\tau_a^j(\mathcal{S}_b)/\epsilon) = \exp(j\partial_a(\mathcal{S}_{b,0}) + O(\epsilon)) \exp(\mathcal{S}_b/\epsilon), \\ \bar{\tau}_a^j(\exp(\mathcal{S}_b/\epsilon)) &= \exp(\bar{\tau}_a^j(\mathcal{S}_b)/\epsilon) = \exp(j\bar{\partial}_a(\mathcal{S}_{b,0}) + O(\epsilon)) \exp(\mathcal{S}_b/\epsilon), \\ \partial_{ja}(\exp(\mathcal{S}_b/\epsilon)) &= (\partial_{ja}(\mathcal{S}_{b,0}) + O(\epsilon)) \exp(\mathcal{S}_b/\epsilon). \end{aligned}$$

### 3.1 Hamilton–Jacobi equations and dispersionless Whitham hierarchies

As  $\epsilon \rightarrow 0$  it follows that

$$\begin{aligned} (\mathcal{L}_a)_+^j(\Psi_b) &= \left( \mathcal{P}_{ja}(e^{\partial_a \mathcal{S}_{b,0}}) + O(\epsilon) \right) \Psi_b, \\ (\bar{\mathcal{L}}_a)_+^j(\Psi_b) &= \left( \bar{\mathcal{P}}_{ja}(e^{\partial_a \mathcal{S}_{b,0}}) + O(\epsilon) \right) \Psi_b, \end{aligned}$$

where  $\mathcal{P}_{ja}$  and  $\bar{\mathcal{P}}_{ja}$  are polynomials

$$\begin{aligned}\mathcal{P}_{j1}(Z) &= Z^j + P_{j1,j-1}Z^{j-1} + \cdots + P_{j1,0}, \\ \mathcal{P}_{ja}(Z) &= P_{ja,j}Z^j + \cdots + P_{ja,1}Z, \end{aligned} \quad a \neq 1$$

$$\begin{aligned}\bar{\mathcal{P}}_{j\bar{1}}(Z) &= \bar{P}_{j\bar{1},j}Z^j + \bar{P}_{j\bar{1},j-1}Z^{j-1} + \cdots + \bar{P}_{j\bar{1},1}Z - (1 - \delta_{1a_0}) \sum_{i=1}^j \bar{P}_{j\bar{1},i}, \\ \bar{\mathcal{P}}_{ja}(Z) &= \bar{P}_{ja,j}Z^j + \cdots + \bar{P}_{ja,1}Z - (1 - \delta_{a1}) \sum_{i=1}^j \bar{P}_{ja,i}, \end{aligned} \quad a \neq \bar{1}.$$

Hence, as  $\epsilon \rightarrow 0$  we get from (46) the following Hamilton–Jacobi type equations

**Proposition 10.** *The following equations holds*

$$\partial_{ja}(\mathcal{S}_{b,0}) = \mathcal{P}_{ja}(e^{\partial_a \mathcal{S}_{b,0}}), \quad (47)$$

$$\partial_{ja}(\mathcal{S}_{b,0}) = \bar{\mathcal{P}}_{ja}(e^{\bar{\partial}_a \mathcal{S}_{b,0}}). \quad (48)$$

Next we show how these equations lead to the two pictures of the Whitham hierarchy described in the Appendix A.

### 3.1.1 KP and Toda dispersionless limits from the Hamilton–Jacobi equations

From the basic equation

$$\frac{\partial \Psi_b}{\partial t_{11}} = (\mathcal{L}_1)_+(\Psi_b),$$

we get the important formula

$$\partial_{t_{11}}(\mathcal{S}_{b,0}) = e^{(\partial_{s_1} - \partial_{s_a})(\mathcal{S}_{b,0})} + q_a, \quad a \neq 1. \quad (49)$$

Where  $q_a$  is an appropriate function defined in terms of derivatives of the leading coefficient of  $\varphi_{1,11}$ . Observe that a family of equations as (49) only occurs for the time  $t_{11}$  and not for the times  $t_{1a}$  with  $a \neq 1$ . This is a consequence of the fact that we have chosen the first row in the matrix wave functions, and we are dealing with the shifts of type  $\tau_a$ .

### The KP-picture dispersionless limit

**Definition 1.** *We introduce the dispersionless Lax functions in the KP picture,  $z_a = z_a(\mathbf{s}, \mathbf{t})$  by the implicit relations*

$$p = \partial_x \mathcal{S}_{a,0}(z_a), \quad x := t_{11},$$

and the corresponding dispersionless Orlov–Schulman functions by

$$m_a := \left. \frac{\partial \mathcal{S}_{a,0}}{\partial z} \right|_{z=z_a}.$$

This definition implies

$$e^{\partial_1 \mathcal{S}_{1,0}} \Big|_{z=z_1} = p - q_{a_0}, \quad e^{\partial_a \mathcal{S}_{a,0}} \Big|_{z=z_a} = \frac{1}{p - q_a}, \quad a \neq 1.$$

The next Proposition exhibits the asymptotic form of these functions

**Proposition 11.** *The dispersionless Lax and Orlov–Schulman functions satisfy*

$$\begin{aligned} z_a^{\text{sg } a} &= \begin{cases} p + \ell_{1,0} + O(p^{-1}), & p \rightarrow \infty, \quad a = 1, \\ \frac{\ell_{a,1}}{p - q_a} + \ell_{a,0} + O(p - q_a), & p \rightarrow q_a \quad a \neq 1, \end{cases} \\ m_a &= (n_0 + \text{sg}(a)s_a)z_a^{-1} + \sum_{j=1}^{\infty} j t_{ja} z_a^{j-1} + z_a^{-1} \begin{cases} \sum_{j=1}^{\infty} \mu_{aj}(p - q_{a'})^{-j}, & a = 1, \\ \sum_{j=1}^{\infty} \mu_{aj}(p - q_a)^j, & a \neq 1. \end{cases} \end{aligned}$$

*Proof.* Particular cases of (36) are

$$\mathcal{M}_a(\psi_a) = z \frac{d\psi_a}{dz}, \quad \mathcal{L}_a(\psi_a) = z^{\text{sg } a} \psi_a,$$

which together with (29) and (33) imply

$$z^{\text{sg } a} = \begin{cases} e^{\partial_l \mathcal{S}_{1,0} + \ell_{1,0} + \ell_{1,-1} e^{-\partial_l \mathcal{S}_{1,0}} + \ell_{1,-2} e^{-2\partial_l \mathcal{S}_{1,0}} + \dots}, & a = 1 \\ \ell_{a,1} e^{\partial_a \mathcal{S}_{a,0} + \ell_{a,0} + \ell_{a,-1} e^{-\partial_a \mathcal{S}_{a,0}} + \ell_{a,-2} e^{-2\partial_a \mathcal{S}_{a,0}} + \dots}, & a \neq 1, \end{cases}$$

$$\frac{\partial \mathcal{S}_{a,0}}{\partial z} = (n_0 + \text{sg}(a)s_a)z^{-1} + \sum_{j=1}^{\infty} j t_{ja} z^{j-1} + z^{-1} \sum_{j=1}^{\infty} \mu_{aj} e^{-j\partial_a \mathcal{S}_{a,0}}$$

and the evaluation at  $z = z_a$  gives the desired result.  $\square$

Therefore for  $a \neq 1$  we have

$$\begin{aligned} (\partial_a(\mathcal{S}_{b,0})) \Big|_{z=z_b} &= -\log(p - q_a), & a \neq 1 \\ (\partial_{ja}(\mathcal{S}_{b,0})) \Big|_{z=z_b} &= \mathcal{P}_{ja} \left( \frac{1}{p - q_a} \right) =: \Omega_{ja}, & a \neq 1 \\ (\partial_{j1}(\mathcal{S}_{b,0})) \Big|_{z=z_b} &= \mathcal{P}_{j1}(p - q_{a'}) =: \Omega_{j1}, & j > 1 \end{aligned}$$

Then we have that

$$d\mathcal{S}_{b,0} = m_b dz_b + p dx - \sum_{a \neq 1} \log(p - q_a) ds_a + \sum'_{j,a} \Omega_{ja} dt_{ja}$$

where the  $\Sigma'$  indicates the sum over the set of indexes  $(j, a)$  where  $j = 1, 2, \dots$  and  $a \in \mathcal{S}$  excluding the case  $j = 1$  and  $a = l$ . Thus the functions  $d\mathcal{S}_{b,0}$  determine a solution of the zero-genus Whitham hierarchy with  $2N$  punctures in the KP picture (see Appendix A).

**The Toda-picture dispersionless limit** We again consider equation (49)

$$\partial_{t_{11}}(\mathcal{S}_{b,0}) = e^{(\partial_{s_1} - \partial_{s_a})(\mathcal{S}_{b,0})} + q_a,$$

which implies

$$e^{-\partial_{a_0} \mathcal{S}_{b,0}} = e^{-\partial_a \mathcal{S}_{b,0}} + Q_a, \quad a, a_0 \neq 1, \quad Q_a := q_a - q_{a_0}$$

**Definition 2.** In the Toda representation the dispersionless Lax function  $z_a = z_a(\mathbf{s}, \mathbf{t})$  is given by the implicit relation

$$p = e^{-\partial_x \mathcal{S}_{b,0}} \Big|_{z=z_b}, \quad x := -\sigma_{a_0},$$

and the dispersionless Orlov-Schulman function by

$$m_a := z \frac{\partial \mathcal{S}_{a,0}}{\partial z} \Big|_{z=z_a}.$$

Observing that

$$e^{\partial_a \mathcal{S}_{b,0}} = \frac{1}{e^{-\partial_{a_0} \mathcal{S}_{b,0}} - Q_a}$$

we conclude

$$e^{\partial_x \mathcal{S}_{1,0}} \Big|_{z=z_1} = p, \quad e^{\partial_{a_0} \mathcal{S}_{a_0,0}} \Big|_{z=z_{a'}} = p^{-1}, \quad e^{\partial_a \mathcal{S}_{a,0}} \Big|_{z=z_a} = \frac{1}{p - Q_a}, \quad a_0 \neq 1 \quad a \neq 1, a_0. \quad (50)$$

Hence, we deduce

**Proposition 12.** *The dispersionless Lax and Orlov–Schulman functions in the Toda-picture dispersionless limit satisfy*

$$z_a^{\text{sg } a} = \begin{cases} p + \ell_{1,0} + O(p^{-1}), & p \rightarrow \infty, & a = 1, \\ \ell_{2,1}p^{-1} + O(1), & p \rightarrow 0, & a = a_0, \\ \frac{\ell_{a,1}}{p - Q_a} + O(1), & p \rightarrow Q_a, & a \neq 1, a_0. \end{cases}$$

$$m_a = (n_0 + \text{sg}(a)s_a) + \sum_{j=1}^{\infty} j t_{ja} z_a^j + \begin{cases} \sum_{j=1}^{\infty} \mu_{1j} p^{-j}, & a = 1, \\ \sum_{j=1}^{\infty} \mu_{aj} p^j, & a = a_0, \\ \sum_{j=1}^{\infty} \mu_{aj} (p - Q_a)^j, & a \neq 1, a_0. \end{cases}$$

*Proof.* Proceed as in the proof of Proposition 11 and use (50).  $\square$

As in the KP case we get now

$$\begin{aligned} (\partial_a(\mathcal{S}_{b,0})) \Big|_{z=z_b} &= -\log(p - Q_a), & a \neq 1, a_0, \\ (\partial_{ja}(\mathcal{S}_{b,0})) \Big|_{z=z_b} &= \mathcal{P}_{ja} \left( \frac{1}{p - Q_a} \right) =: \Omega_{ja}, & a \neq 1, a_0, \\ (\partial_{j1}(\mathcal{S}_{b,0})) \Big|_{z=z_b} &= \mathcal{P}_{j1}(p) =: \Omega_{j1}, \\ (\partial_{ja_0}(\mathcal{S}_{b,0})) \Big|_{z=z_b} &= \mathcal{P}_{ja_0}(p^{-1}) =: \Omega_{ja_0}. \end{aligned}$$

Hence we have that

$$d\mathcal{S}_{b,0} = m_b d \log z_b + \log p dx - \sum_{a \neq 1, a'} \log(p - Q_a) ds_a + \sum_{j \geq 1, a \in \mathcal{S}} \Omega_{ja} dt_{ja},$$

and therefore the functions  $\mathcal{S}_{b,0}$  determine a solution of the zero-genus Whitham hierarchy with  $2N$  punctures in the Toda picture (see Appendix A).

**An alternative Toda-picture dispersionless limit** From the basic equation

$$\frac{\partial \Psi_b}{\partial t_{1\bar{1}}} = (\bar{\mathcal{L}}_{\bar{1}})_+(\Psi_b),$$

we deduce

$$\partial_{1\bar{1}} \mathcal{S}_{b,0} = r_1 e^{(\partial_{s_{\bar{1}}} - \partial_{s_1})(\mathcal{S}_{b,0})} = r_a (e^{(\partial_{s_{\bar{1}}} - \partial_{s_a})(\mathcal{S}_{b,0})} - 1), \quad a \neq 1, \bar{1}$$

for some functions  $r_a$ . Hence,

$$e^{\bar{\partial}_a \mathcal{S}_{b,0}} = \frac{e^{\bar{\partial}_1 \mathcal{S}_{b,0}}}{e^{\bar{\partial}_1 \mathcal{S}_{b,0}} + \rho_a}, \quad \rho_a := \frac{r_a}{r_1}.$$

Now, we take  $a_0 = 1$  and define

**Definition 3.** *The dispersionless Lax functions  $z_a$  are defined by the implicit relation*

$$\partial_x \mathcal{S}_{a,0} \Big|_{z=z_a} = \log p, \quad x = -\bar{\sigma}_1$$

*while the dispersionless Orlov–Schulman functions are defined by*

$$m_a := z_a \frac{\partial \mathcal{S}_{a,0}}{\partial z} \Big|_{z=z_a}.$$

Observe that

$$e^{\partial_1 \mathcal{S}_{1,0}} \Big|_{z=z_1} = p, \quad e^{\partial_{\bar{1}} \mathcal{S}_{\bar{1},0}} \Big|_{z=z_{\bar{1}}} = p^{-1}, \quad e^{\partial_a \mathcal{S}_{a,0}} \Big|_{z=z_a} = \frac{p}{p + \rho_a}, \quad a \neq 1, \bar{1}.$$

**Proposition 13.** *The dispersionless Lax and Orlov–Schulman functions are of the following form*

$$z_a^{\text{sg } a} = \begin{cases} p + \bar{\ell}_{1,0} + O(p^{-1}), & p \rightarrow \infty, \quad a = 1 \\ \bar{\ell}_{1,1} p^{-1} + \bar{\ell}_{1,0} + O(p), & p \rightarrow 0, \quad a = \bar{1} \\ \bar{\ell}_{a,1} \frac{p}{p + \rho_a} + \bar{\ell}_{a,0} + O\left(\left(\frac{p}{p + \rho_a}\right)^{-1}\right), & p \rightarrow -\rho_a, \quad a \neq 1, \end{cases}$$

$$m_a = (n_0 + \text{sg}(a)s_a) + \sum_{j=1}^{\infty} j t_{ja} \bar{z}_a^{j-1} + \begin{cases} \sum_{j=1}^{\infty} \mu_{aj} p^{-j}, & a = 1, \\ \sum_{j=1}^{\infty} \mu_{aj} p^j, & a = \bar{1}, \\ \sum_{j=1}^{\infty} \mu_{aj} \left(\frac{p}{p + \rho_a}\right)^j, & a \neq 1, \bar{1}. \end{cases}$$

*Proof.* Particular cases of (36) are

$$\bar{\mathcal{M}}_a(\Psi_a) = z \frac{d\Psi_a}{dz}, \quad \bar{\mathcal{L}}_a(\Psi_a) = z^{\text{sg } a} \Psi_a,$$

which together with (29) and (33) imply

$$z^{\text{sg } a} = \begin{cases} e^{\bar{\partial}_1 \mathcal{S}_{1,0}} + \bar{\ell}_{1,0} + \bar{\ell}_{1,-1} e^{-\bar{\partial}_1 \mathcal{S}_{1,0}} + \bar{\ell}_{1,-2} e^{-2\bar{\partial}_1 \mathcal{S}_{1,0}} + \dots, & a = 1, \\ \bar{\ell}_{a,1} e^{\bar{\partial}_a \mathcal{S}_{a,0}} + \bar{\ell}_{a,0} + \bar{\ell}_{a,-1} e^{-\bar{\partial}_a \mathcal{S}_{a,0}} + \bar{\ell}_{a,-2} e^{-2\bar{\partial}_a \mathcal{S}_{a,0}} + \dots, & a \neq 1, \end{cases}$$

$$z \frac{\partial \mathcal{S}_{a,0}}{\partial z} = (n_0 + \text{sg}(a)s_a) + \sum_{j=1}^{\infty} j t_{ja} z^j + \sum_{j=1}^{\infty} \bar{\mu}_{aj} e^{-j \bar{\partial}_a \mathcal{S}_{a,0}}.$$

and the evaluation of these expressions at  $z = z_a$  gives the result.  $\square$

Therefore,

$$\begin{aligned} \left. \partial_x(\mathcal{S}_{b,0}) \right|_{z=z_b} &= \log p, \\ \left. \partial_a(\mathcal{S}_{b,0}) \right|_{z=z_b} &= \log \left( \frac{p}{p + \rho_a} \right), & a \neq 1, \bar{1}, \\ \left. (\partial_{ja}(\mathcal{S}_{b,0})) \right|_{z=z_b} &= \bar{\mathcal{P}}_{ja} \left( \frac{p}{p + \rho_a} \right) =: \Omega_{ja}, & a \neq 1, \bar{1}, \\ \left. \partial_{j\bar{1}}(\mathcal{S}_{b,0}) \right|_{z=z_b} &= \bar{\mathcal{P}}_{j\bar{1}}(p^{-1}) =: \Omega_{j\bar{1}}, \\ \left. (\partial_{j1}(\mathcal{S}_{b,0})) \right|_{z=z_b} &= \bar{\mathcal{P}}_{j1}(p) =: \Omega_{j1}, & j > 1. \end{aligned}$$

In this way we have

$$d\mathcal{S}_{b,0} = m_b d \log z_b + \log(p) dx + \sum_{a \neq \bar{1}, 1} \log \left( \frac{p}{p + \rho_a} \right) ds_a + \sum_{j \geq 1, a \in \mathcal{S}} \Omega_{ja} dt_{ja}.$$

As we will show at the end of this section the functions  $\mathcal{S}_{b,0}$  determine a solution of the zero-genus Whitham hierarchy with  $2N$  punctures in the Toda picture.

### 3.2 The dispersionless limits of the string equations

Let us consider operators of the form

$$F_a(\mathcal{M}_a, \mathcal{L}_a) = \sum_{i \geq 0, j \in \mathbb{Z}} F_{aij} \mathcal{M}_a^i \mathcal{L}_a^j, \quad F_a(\bar{\mathcal{M}}_a, \bar{\mathcal{L}}_a) = \sum_{i \geq 0, j \in \mathbb{Z}} F_{aij} \bar{\mathcal{M}}_a^i \bar{\mathcal{L}}_a^j.$$

In order to formulate their dispersionless limits it is convenient to assume that the coefficients satisfy as  $\epsilon \rightarrow 0$  that

$$F_{aij} = F_{aij,0} \epsilon^i + O(\epsilon^{i+1}).$$

Recalling (36) and observing that

$$\left(z \frac{d}{dz}\right)^i = z^i \frac{d^i}{dz^i} + \sum_{i'=2}^i \binom{i'}{2} z^{i'-1} \frac{d^{i'-1}}{dz^{i'-1}},$$

$$\frac{\partial^i \Psi_a}{\partial z^i} = \left(\epsilon^{-i} \left(\frac{\partial \mathcal{S}_{a,0}}{\partial z}\right)^i + O(\epsilon^{-i+1})\right) \Psi_a$$

we get

$$\overleftarrow{(\Psi_a) F_a \left(z \frac{d}{dz}, z^{\text{sg } a}\right)} = \left[ \sum_{i \geq 0, j \in \mathbb{Z}} F_{aij,0} z^{\text{sg}(a)j} \left(\frac{\partial \mathcal{S}_{a,0}}{\partial z}\right)^i + O(\epsilon) \right] \Psi_a.$$

Hence,

$$F_{a,0}(z_a, m_a) := \left[ \lim_{\epsilon \rightarrow 0} \overleftarrow{(\Psi_a) F_a \left(z \frac{d}{dz}, z^{\text{sg } a}\right)} \Psi_a^{-1} \right]_{z=z_a} = \begin{cases} \sum_{i \geq 0, j \in \mathbb{Z}} F_{aij,0} z_a^{i+\text{sg}(a)j} m_a^i, & \text{KP,} \\ \sum_{i \geq 0, j \in \mathbb{Z}} F_{aij,0} z_a^{\text{sg}(a)j} m_a^i, & \text{Toda.} \end{cases}$$

We define

$$F_{a,0+} := \begin{cases} \left[ \lim_{\epsilon \rightarrow 0} \frac{F_a(\mathcal{M}_a, \mathcal{L}_a) + (\Psi_a)}{\Psi_a} \right]_{z=z_a}, & \text{unbared cases} \\ \left[ \lim_{\epsilon \rightarrow 0} \frac{F_a(\bar{\mathcal{M}}_a, \bar{\mathcal{L}}_a) + (\Psi_a)}{\Psi_a} \right]_{z=z_a}, & \text{bared cases} \end{cases}$$

**Proposition 14.** *Given*

$$F_a(\mathcal{M}_a, \mathcal{L}_a) = \sum_{j \in \mathbb{Z}} f_{aj} \mathcal{T}_a^j, \quad \text{or} \quad F_a(\bar{\mathcal{M}}_a, \bar{\mathcal{L}}_a) = \sum_{j \in \mathbb{Z}} \bar{f}_{aj} \bar{\mathcal{T}}_a^j,$$

with  $f_{ai} = f_{ai|0} + O(\epsilon)$  as  $\epsilon \rightarrow 0$ , their dispersionless limits are

$$F_{a,0} = \begin{cases} \begin{cases} \sum_{j \in \mathbb{Z}} \frac{f_{aj|0}}{(p - q_a)^j}, & a \neq 1, \\ \sum_{j \in \mathbb{Z}} f_{1j|0} (p - q_{a'})^j & a = 1, \end{cases} & \text{KP,} \\ \begin{cases} \sum_{j \in \mathbb{Z}} \frac{f_{aj|0}}{(p - Q_a)^j}, & a \neq 1, a', \\ \sum_{j \in \mathbb{Z}} f_{1j|0} p^j, & a = 1, \\ \sum_{j \in \mathbb{Z}} f_{a'j|0} p^{-j}, & a = a', \end{cases} & \text{Toda,} \\ \begin{cases} \sum_{j \in \mathbb{Z}} f_{aj|0} \left(\frac{p}{p + \rho_a}\right)^j, & a \neq 1, \bar{1}, \\ \sum_{j \in \mathbb{Z}} f_{1j|0} p^j, & a = 1, \\ \sum_{j \in \mathbb{Z}} f_{\bar{1}j|0} p^{-j}, & a = \bar{1}, \end{cases} & \text{Alternative Toda.} \end{cases}$$

Moreover,

$$F_{a,0+} = \begin{cases} \begin{cases} \sum_{j > 0} \frac{f_{aj|0}}{(p - q_a)^j}, & a \neq 1, \\ \sum_{j \geq 0} f_{1j|0} (p - q_{a'})^j & a = 1, \end{cases} & \text{KP,} \\ \begin{cases} \sum_{j > 0} \frac{f_{aj|0}}{(p - Q_a)^j}, & a \neq 1, a', \\ \sum_{j \geq 0} f_{1j|0} p^j, & a = 1, \\ \sum_{j > 0} f_{a'j|0} p^{-j}, & a = a', \end{cases} & \text{Toda,} \\ \begin{cases} \sum_{j > 0} f_{aj|0} \left(\left(\frac{p}{p + \rho_a}\right)^j - 1\right), & a \neq 1, \bar{1}, \\ \sum_{j \geq 0} f_{1j|0} p^j, & a = 1, \\ \sum_{j > 0} f_{\bar{1}j|0} p^{-j}, & a = \bar{1}, \end{cases} & \text{Alternative Toda.} \end{cases}$$

In particular

$$\Omega_{ja} = \begin{cases} (z_a^{\text{sg}(a)j})_+, & \text{KP, Toda,} \\ (z_a^{\text{sg}(a)j})_+, & \text{Alternative Toda.} \end{cases}$$

*Proof.* The formulae follow from the identity

$$\begin{aligned} \overleftarrow{(\Psi_a)F_a\left(z\frac{d}{dz}, z^{\text{sg } a}\right)} &= F_a(\mathcal{M}_a, \mathcal{L}_a)(\Psi_a) = \sum_{j \in \mathbb{Z}} f_{aj} e^{j\partial_a \mathcal{S}_{a,0} + O(\epsilon)} \Psi_a \\ &= F_a(\bar{\mathcal{M}}_a, \bar{\mathcal{L}}_a)(\Psi_a) = \sum_{j \in \mathbb{Z}} f_{aj} e^{j\bar{\partial}_a \mathcal{S}_{a,0} + O(\epsilon)} \Psi_a. \end{aligned}$$

□

As a consequence

**Proposition 15.** *If the string equations (44) hold, their corresponding dispersionless limits*

$$F_{a,0}(z_a, m_a) = \sum_{b \in \mathcal{S}} F_{b,0+}, \quad \forall a \in \mathcal{S} \quad (51)$$

are satisfied.

*Proof.* It follows from Proposition 8. □

These dispersionless string equations are of the type considered in [8] for the dispersionless Whitham hierarchy. Moreover, given a decomposition  $\mathcal{S} = \mathcal{I} \cup \mathcal{J}$  into two disjoint subsets, we may take

$$P_{a,0} = \begin{cases} z_a^{\ell_a}, & a \in \mathcal{I}, \\ -\frac{m_a}{\ell_a z_a^{\ell_a-1}}, & a \in \mathcal{J}, \end{cases} \quad Q_{a,0} = \begin{cases} \frac{m_a}{\ell_a z_a^{\ell_a-1}}, & a \in \mathcal{I}, \\ z_a^{\ell_a}, & a \in \mathcal{J}. \end{cases}$$

The corresponding dispersive string equations are

$$\begin{aligned} E_{11}\left(\sum_{k \in (\mathcal{I} \cap \mathbb{S})} L^{\ell_k} C_{kk} - \sum_{k \in (\mathcal{J} \cap \mathbb{S})} \ell_k^{-1} M L^{-\ell_k+1} C_{kk}\right) &= E_{11}\left(\sum_{k \in (\mathcal{I} \cap \bar{\mathbb{S}})} \bar{L}^{-\ell_k} \bar{C}_{kk} - \sum_{k \in (\mathcal{J} \cap \bar{\mathbb{S}})} \ell_k^{-1} \bar{M} \bar{L}^{\ell_k-1} \bar{C}_{kk}\right), \\ E_{11}\left(\sum_{k \in (\mathcal{I} \cap \mathbb{S})} \ell_k^{-1} M L^{-\ell_k+1} C_{kk} + \sum_{k \in (\mathcal{J} \cap \mathbb{S})} L^{\ell_k} C_{kk}\right) &= E_{11}\left(\sum_{k \in (\mathcal{I} \cap \bar{\mathbb{S}})} \ell_k^{-1} \bar{M} \bar{L}^{\ell_k-1} \bar{C}_{kk} + \sum_{k \in (\mathcal{J} \cap \bar{\mathbb{S}})} \bar{L}^{-\ell_k} \bar{C}_{kk}\right). \end{aligned}$$

If  $\mathcal{J} = \emptyset$  we get

$$\begin{aligned} E_{11} \sum_{k=1}^N L^{\ell_k} C_{kk} &= E_{11} \sum_{k=1}^N \bar{L}^{-\ell_k} \bar{C}_{kk}, \\ E_{11} \sum_{k=1}^N \ell_k^{-1} M L^{-\ell_k+1} C_{kk} &= E_{11} \sum_{k=1}^N \ell_k^{-1} \bar{M} \bar{L}^{\ell_k-1} \bar{C}_{kk}. \end{aligned}$$

For positive integers  $\ell_a$  the dispersionless limits of these dispersive string equations describe the algebraic orbits of the genus 0 Whitham hierarchy [24]. The first of these dispersive string equations gives the multigraded reduction as discussed in [1].

We return to the equivalence of the alternative Toda and Toda pictures. First we notice that within alternative Toda picture we have Laurent expansions in  $Z_a$

$$Z_a := \frac{p}{p + \rho_a} = 1 - \rho_a \zeta_a, \quad \zeta_a := \frac{1}{p + \rho_a}.$$

The functions  $Z_a^j$  are singular at  $p = -\rho_a$  and  $\lim_{p \rightarrow \infty} Z_a^j = 1$ , thus the linear combinations of factors  $Z_a^j - 1$  which appear in the construction of the  $\Omega_{ja}$ , lead to singular functions in  $p = -\rho_a$  normalized to 0 at infinity. Hence, if we express  $z_a$  as Laurent series in  $\zeta_a$ , the function  $\Omega_{ja}$  is just the singular part corresponding to the



projection to power series in  $\zeta_a$  with non constant term. Thus, we recover the Toda picture of the genus 0 Whitham hierarchy (see Appendix A).

Given operators  $P_a$  and  $Q_a$  as in Proposition 3 we get for the commutator

$$\begin{aligned} (\Psi_a)^{\overleftarrow{[P_a, Q_a]}} \left( z \frac{d}{dz}, z^{\text{sg } a} \right) &= (\Psi_a) \left[ P_a \left( z \frac{d}{dz}, z^{\text{sg } a} \right), Q_a \left( z \frac{d}{dz}, z^{\text{sg } a} \right) \right] \\ &= \left[ \sum_{\substack{i_1, i_2 \geq 0 \\ j_1, j_2 \in \mathbb{Z}}} P_{ai_1j_1, 0} Q_{ai_2j_2, 0} \text{sg}(a)(i_2j_1 - i_1j_2) z^{\text{sg}(a)(j_1+j_2)+i_1+i_2-1} \left( \frac{\partial \mathcal{S}_{a,0}}{\partial} \right)^{i_1+i_2-1} \epsilon + O(\epsilon^2) \right] \Psi_a \end{aligned}$$

so that for the KP picture we find

$$\begin{aligned} \left[ \lim_{\epsilon \rightarrow 0} (\Psi_a)^{\overleftarrow{\epsilon^{-1}[P_a, Q_a]}} \left( z \frac{d}{dz}, z \right) \Psi_a^{-1} \right]_{z=z_a} &= \sum_{\substack{i_1, i_2 \geq 0 \\ j_1, j_2 \in \mathbb{Z}}} P_{ai_1j_1, 0} Q_{ai_2j_2, 0} \text{sg}(a)(i_2j_1 - i_1j_2) z_a^{\text{sg}(a)(j_1+j_2)+i_1+i_2-1} m_a^{i_1+i_2-1} \\ &= \{P_{0,a}, Q_{0,a}\}_0, \end{aligned}$$

while for the Toda picture we have

$$\begin{aligned} \left[ \lim_{\epsilon \rightarrow 0} (\Psi_a)^{\overleftarrow{\epsilon^{-1}[P_a, Q_a]}} \left( z \frac{d}{dz}, z \right) \Psi_a^{-1} \right]_{z=z_a} &= \sum_{\substack{i_1, i_2 \geq 0 \\ j_1, j_2 \in \mathbb{Z}}} P_{ai_1j_1, 0} Q_{ai_2j_2, 0} \text{sg}(a)(i_2j_1 - i_1j_2) z_a^{\text{sg}(a)(j_1+j_2)+i_1+i_2-1} m_a^{i_1+i_2-1} \\ &= \{P_{0,a}, Q_{0,a}\}_1. \end{aligned}$$

Thus,

$$[P_a, Q_a] = \epsilon \quad \implies \quad \{P_{a,0}, Q_{a,0}\}_0 = 1 \text{ or } \{P_{a,0}, Q_{a,0}\}_1 = 1.$$

## Appendix A: Whitham hierarchies in the zero genus case

The zero-genus Whitham hierarchies [24] are systems of flows on a *phase space*  $\widehat{M}_0$  of data associated to algebraic Riemann surfaces of genus 0. The points of  $\widehat{M}_0$  are given  $(\Gamma, Q_a, z_a^{-1})$ , where  $\Gamma$  is an algebraic Riemann surface of genus 0,  $Q_a$  are  $N$  points (punctures) of  $\Gamma$  and  $z_a^{-1}$  denote local coordinates around each  $Q_a$  such that  $z_a^{-1}(Q_a) = 0$ . In order to formulate Whitham flows on  $\widehat{M}_0$  it is convenient to introduce a meromorphic function  $p = p(Q)$  on  $\Gamma$  such that the local coordinates have asymptotic expansions of the form

$$z_a = \begin{cases} p + \ell_{1,0} + \sum_{n=1}^{\infty} \frac{\ell_{1,-n}}{p^n}, & a = 1, \\ \frac{\ell_{a,1}}{p - q_a} + \sum_{n=0}^{\infty} \ell_{a,-n} (p - q_a)^n, & a = 2, \dots, N, \end{cases} \quad (52)$$

where  $p(Q_a) = q_a$  with  $q_1 = \infty$ . In general the points of the phase space  $\widehat{M}_0$  are characterized by an infinite number of parameters  $\mathbf{w} := (w_i)$  of the set  $(q_a, \ell_{a,n})$ . However, under appropriate reduction conditions on the form of  $\Gamma$  only a finite number of these parameters are independent and constitute a coordinate system for  $\widehat{M}_0$ .

### Example: Algebraic orbits

If we restrict to zero-genus Riemann surfaces  $\Gamma$  of the form

$$\lambda = p^{n_1} + \sum_{n=0}^{n_1-1} u_{1n} p^n + \sum_{i=2}^N \sum_{n=1}^{n_q} \frac{u_{in}}{(p - q_i)^n}, \quad (53)$$

we may take  $Q_a = (\lambda_a, p_a) = (\infty, q_a)$  ( $a = 1, \dots, q$ ), with corresponding local coordinates given by

$$z_a = \lambda^{1/n_a}.$$

The function  $p(\lambda, p) = p$  is meromorphic on  $\Gamma$  and the local coordinates have asymptotic expansions of the form (52). In this case the points of the phase space  $\widehat{M}_0$  are characterized by the parameters  $\mathbf{w} = (\{q_a\}_{a=2}^N, \{u_{1n}\}_{n=0}^{n_1}, \dots, \{u_{Nn}\}_{n=1}^{n_N})$

The Whitham flows  $\mathbf{w}(\mathbf{t}) = (w_i(\mathbf{t}))$  are introduced through sets  $\mathbf{\Omega} := \{\Omega_A(\mathbf{w}, p)\}$  of functions, with meromorphic differentials  $\partial_p \Omega_A(\mathbf{w}, p) \mathrm{d}p$ , which satisfy the conditions:

1. One of the functions  $\Omega_{A_0}$  is independent of the data  $\mathbf{w}$ .
2. There exist local functions  $S_a(\mathbf{t}, z_a)$  around the punctures satisfying

$$\partial_A S_a(\mathbf{t}, z_a) = \Omega_A(\mathbf{w}(\mathbf{t}), p(\mathbf{t}, z_a)). \quad (54)$$

Here  $\partial_A := \partial/\partial t_A$  and  $\mathbf{t}$  denotes the set of flow parameters  $t_A$ .

The first condition only demands to include a function of the form  $\Omega(p)$  in  $\mathbf{\Omega}$ . On the other hand, it is obvious that the second condition is satisfied if and only if the following Zakharov-Shabat equations are satisfied

$$\partial_B \Omega_A - \partial_A \Omega_B + \{\Omega_A, \Omega_B\} = 0, \quad (55)$$

where  $\partial_t := \sum_i \partial_t w_i \partial_{w_i}$  for  $t = t_A, t_B$ , and  $\{, \}$  denotes the Poisson bracket

$$\{F, G\} := \omega(p) \left( \partial_p F \partial_x G - \partial_x F \partial_p G \right), \quad \omega(p) := (\partial_p \Omega_{A_0}(p))^{-1}, \quad x := t_{A_0}.$$

We may write (54) as

$$\mathrm{d} S_a = m_a \mathrm{d} \Omega_{A_0}(z_a) + \sum_A \Omega_A \mathrm{d} t_A, \quad m_a := \omega(z_a)^{-1} \frac{\partial S_a}{\partial z_a}, \quad (56)$$

which implies

$$\mathrm{d} \Omega_{A_0}(z_a) \wedge \mathrm{d} m_a = \sum_A \mathrm{d} \Omega_A \wedge \mathrm{d} t_A, \quad (57)$$

and by equating the coefficients of  $\mathrm{d} p \wedge \mathrm{d} x$  in both members of (57) yields

$$\{z_a, m_a\} = \omega(z_a). \quad (58)$$

Moreover, if we identify the coefficients of  $\mathrm{d} p \wedge \mathrm{d} t_A$  and  $\mathrm{d} x \wedge \mathrm{d} t_A$  in (57) we get

$$\begin{cases} \partial_p z_a \partial_A m_a - \partial_A z_a \partial_p m_a = \omega(z_a) \partial_p \Omega_A, \\ \partial_x z_a \partial_A m_a - \partial_A z_a \partial_x m_a = \omega(z_a) \partial_x \Omega_A, \end{cases}$$

so that taking (58) into account we deduce the system of Lax equations

$$\partial_A z_a = \{\Omega_A, z_a\}, \quad \partial_A m_a = \{\Omega_A, m_a\}. \quad (59)$$

As it was shown in [?]-[27] important classes of solutions of the zero-genus Whitham hierarchy can be obtained from systems of canonical pairs of constrains (string equations) of the form

$$\begin{cases} P_1(z_1, m_1) = P_2(z_2, m_2) = \dots = P_N(z_N, m_N), \\ Q_1(z_1, m_1) = Q_2(z_2, m_2) = \dots = Q_N(z_N, m_N), \end{cases} \quad (60)$$

where  $(P_a, Q_a)$  are  $N$  pairs of canonically conjugate functions

$$\{P_a(p, x), Q_a(p, q)\} = \omega(p). \quad (61)$$

In particular this type of methods applies for finding solutions for algebraic orbits. Indeed these solutions are associated to string equations generated by

$$P_a(p, x) = p^{n_a}, \quad Q_a(p, x) = \frac{x}{n_a p^{n_a-1}} + f_a(p). \quad (62)$$

## The KP picture

The KP picture of the zero-genus Whitham hierarchy with  $N$  punctures [24] is formulated by assuming  $\ell_{1,0} \equiv 0$  in the asymptotic expansions (52) and by taking the following functions  $\Omega_A$

$$\Omega_{na} := \begin{cases} (z_a^n)_{(a,+)}, & n \geq 1, \\ -\log(p - q_a), & n = 0, \quad a = 2, \dots, N. \end{cases} \quad (63)$$

Here  $(\cdot)_{(a,+)}$  stand for the projectors on the subspaces generated by  $\{p^n\}_{n=0}^\infty$  (case  $a = 1$ ) and  $\{(p - q_a)^{-n}\}_{n=1}^\infty$  (cases  $a \geq 2$ ). In this case

$$A_0 = (1, 1), \quad x = t_{1,1}, \quad \Omega_{A_0} = p,$$

and the Poisson bracket is given by

$$\{F, G\} := \partial_p F \partial_x G - \partial_x F \partial_p G.$$

The functions  $\Omega_A$  satisfy the compatibility conditions (55) so that there exist functions  $S_a$  such that

$$dS_a = m_a dz_a + p dx - \sum_{a \neq 1} \log(p - q_a) dt_{0a} + \sum_{a=1}^N \sum_{n \geq 1} \Omega_{na} dt_{na}. \quad (64)$$

## The Toda picture

A simple redefinition of the meromorphic function  $p(Q)$  used to define the KP flows of the Whitham hierarchy with  $N$  punctures supplies a different picture (the Toda picture) of the hierarchy. Indeed if we set

$$p_{\text{Toda}} = p_{\text{KP}} - q_{a_0},$$

for a given index  $a_0$ , then now  $u_{1,1} = q_{a_0}$  and we may take

$$A_0 := (0, a_0), \quad x := -t_{0,a_0}, \quad \Omega_{A_0} := -\log p.$$

Thus the Poisson bracket is given by

$$\{F, G\} := p (\partial_p F \partial_x G - \partial_x F \partial_p G),$$

and the functions  $S_a$  satisfy

$$dS_a = m_a d \log z_a + \log p dx - \sum_{a \neq 1, a_0} \log(p - q_a) dt_{0a} + \sum_{a=1}^N \sum_{n \geq 1} \Omega_{na} dt_{na}. \quad (65)$$

## Appendix B: Proof of Proposition (6)

The proof of Proposition (6) requires the following Lemma.

**Lemma 1.** 1. Given  $T = \sum_{j \in \mathbb{Z}} c_j \tau_a^j \in \mathfrak{t}_a$ ,  $a \neq 1$ , then

$$\begin{cases} T(E_{11}W) = T_{>}(W_{1k})E_{1k} + T_{\leq}(W_{11})E_1 + \mathfrak{g}_-W_0, \\ T(E_{11}\bar{W}) = \mathfrak{g}_+\bar{W}_0, \end{cases} \quad a = k \neq 1, \quad (66)$$

$$\begin{cases} T(E_{11}W) = T_{\leq}(W_{11})E_{11} + \mathfrak{g}_-W_0, \\ T(E_{11}\bar{W}) = T_{>}(\bar{W}_{1k})E_{1k} + \mathfrak{g}_+\bar{W}_0, \end{cases} \quad a = \bar{k}, \quad (67)$$

2. Given  $T = \sum_{j \in \mathbb{Z}} c_j \tau_l^j \in \mathfrak{t}_1$ , then

$$\begin{cases} T(E_{11}W) = T_{\geq}(W_{11})E_{11} + T_{<}(W_{1l_0})E_{1l_0} + \mathfrak{g}_-W_0, \\ T(E_{11}\bar{W}) = \mathfrak{g}_+\bar{W}_0, \end{cases} \quad a_0 = l_0 \neq 1, \quad (68)$$

$$\begin{cases} T(E_{11}W) = T_{\geq}(W_{11})E_{11} + \mathfrak{g}_-W_0, \\ T(E_{11}\bar{W}) = T_{<}(\bar{W}_{1l_0})E_{1l_0} + \mathfrak{g}_+\bar{W}_0, \end{cases} \quad a_0 = \bar{l}_0, \quad (69)$$

3. Given  $T = \sum_{j \in \mathbb{Z}} c_j (\bar{\tau}_a^j - 1) + c_0 \in \bar{\mathfrak{t}}_a$ ,  $a \neq \bar{1}, 1$ , then

$$\begin{cases} T(E_{11}W) = T_{>}(W_{1k})E_{1k} + c_0 W_{11}E_{11} + \mathfrak{g}_- W_0, \\ T(E_{11}\bar{W}) = T_{<}(\bar{W}_{11})E_{11} + \mathfrak{g}_+ \bar{W}_0, \end{cases} \quad a = k \neq 1, \quad (70)$$

$$\begin{cases} T(E_{11}W) = c_0 W_{11}E_{11} + \mathfrak{g}_- W_0, \\ T(E_{11}\bar{W}) = T_{>}(\bar{W}_{lk})E_{lk} + T_{<}(\bar{W}_{11})E_{11} + \mathfrak{g}_+ \bar{W}_0, \end{cases} \quad a = \bar{k} \neq \bar{1}, \quad (71)$$

4. Given  $T = \sum_{j \in \mathbb{Z}} c_j \bar{\tau}_1^j \in \bar{\mathfrak{t}}_1$ , then

$$\begin{cases} T(E_{11}W) = T_{\geq}(W_{11})E_{11} + \mathfrak{g}_- W_0, \\ T(E_{11}\bar{W}) = T_{<}(\bar{W}_{11})E_{11} + \mathfrak{g}_+ \bar{W}_0, \end{cases} \quad (72)$$

5. Given  $T = \sum_{j \in \mathbb{Z}} c_j (\bar{\tau}_1^j - 1) + c_0 \in \bar{\mathfrak{t}}_1$ , with  $a_0 \neq 1$ , then

$$\begin{cases} T(E_{11}W) = T_{<}(W_{l_0})E_{l_0} + c_0 W_{ll}E_{11} + \mathfrak{g}_- W_0, \\ T(E_{11}\bar{W}) = T_{>}(\bar{W}_{11}) + \mathfrak{g}_+ \bar{W}_0, \end{cases} \quad a_0 = l_0 \neq 1, \quad (73)$$

$$\begin{cases} T(E_{11}W) = c_0 W_{11}E_{11} + \mathfrak{g}_- W_0, \\ T(E_{11}\bar{W}) = T_{>}(\bar{W}_{11})E_{11} + T_{<}(\bar{W}_{l_0})E_{l_0} + \mathfrak{g}_+ \bar{W}_0, \end{cases} \quad a_0 = \bar{l}_0, \quad (74)$$

6. Given  $T = \sum_{j \in \mathbb{Z}} c_j \bar{\tau}_1^j \in \bar{\mathfrak{t}}_1$  with  $a_0 = 1$ , then

$$\begin{cases} T(E_{11}W) = T_{\leq}(W_{11})E_{11} + \mathfrak{g}_- W_0, \\ T(E_{11}\bar{W}) = T_{>}(\bar{W}_{11})E_{11} + \mathfrak{g}_+ \bar{W}_0. \end{cases} \quad (75)$$

*Proof.* We only prove 1) since the others relations are proven similarly. From (66) observe that

$$\begin{aligned} \tau_k^j(E_{11}W) &= \tau_k^j(E_{11}S)\tau_k^j(W_0) = \tau_k^j(E_{11}S)(E_{kk}\Lambda^j + \mathbb{I}_N - E_{kk} - E_{11} + E_{11}\Lambda^{-j})W_0 \\ &= (\tau_k^j(S_{1k})E_{1k}\Lambda^j + E_{11}\tau_k^j(S)(\mathbb{I}_N - E_{kk} - E_{11}) + \tau_k^j(S_{11})E_{11}\Lambda^{-j})W_0, \\ \tau_k^j(E_{11}\bar{W}) &= \tau_k^j(E_{11}\bar{S})\bar{W}_0 \end{aligned}$$

and therefore

$$\begin{aligned} \tau_k^j(E_{11}W)E_{k'k'} &\in \mathfrak{g}_- W_0, & k' \neq k, 1, \\ \tau_k^j(E_{11}W)E_{kk} &= \tau_k^j(S_{1k})E_{1k}\Lambda^j W_0 \in \mathfrak{g}_- W_0 & \text{if } j \leq 0, \\ \tau_k^j(E_{11}W)E_{11} &= \tau_k^j(S_{11})E_{11}\Lambda^{-j} W_0 \in \mathfrak{g}_- W_0 & \text{if } j > 0, \\ \tau_k^j(E_{11}\bar{W}) &\in \mathfrak{g}_+ \bar{W}_0. \end{aligned}$$

Now, we check (67). Notice that

$$\begin{aligned} \tau_k^j(E_{11}W) &= \tau_k^j(E_{11}S)\tau_k^j(W_0) = \tau_k^j(E_{11}S)(\mathbb{I}_N - E_{11} + E_{11}\Lambda^{-j})W_0 \\ &= (E_{11}\tau_k^j(S)(\mathbb{I}_N - E_{11}) + \tau_k^j(S_{11})E_{11}\Lambda^{-j})W_0, \\ \tau_k^j(E_{11}\bar{W}) &= \tau_k^j(E_{11}\bar{S})\tau_k^j(\bar{W}_0) = \tau_k^j(E_{11}\bar{S})(E_{kk}\Lambda^{-j} + \mathbb{I}_N - E_{kk})\bar{W}_0 \\ &= (E_{11}\tau_k^j(\bar{S})(\mathbb{I}_N - E_{kk}) + \tau_k^j(\bar{S}_{1k})E_{1k}\Lambda^{-j})\bar{W}_0, \end{aligned}$$

and therefore

$$\begin{aligned} \tau_k^j(E_{11}W)E_{k'k'} &\in \mathfrak{g}_- W_0, & k' \neq 1, \\ \tau_k^j(E_{11}W)E_{ll} &= \tau_k^j(S_{ll})E_{11}\Lambda^{-j} W_0 \in \mathfrak{g}_- W_0, & \text{if } j > 0, \\ \tau_k^j(E_{11}\bar{W})E_{k'k'} &\in \mathfrak{g}_+ \bar{W}_0, & k' \neq k, \\ \tau_k^j(E_{11}\bar{W})E_{kk} &= \tau_k^j(\bar{S}_{1k})\Lambda^{-j}\bar{W}_0 \in \mathfrak{g}_+ \bar{W}_0, & \text{if } j \leq 0, \end{aligned}$$

□

*Proof of Proposition 6.* The proof of these results relies on the previous Lemma 1 and Propositions 2, 5. Let us go into details. We first consider (40). From (66) we find for  $k \neq 1$

$$F(\mathcal{M}_k, \mathcal{L}_k)_{>}(E_{11}W) = F(\mathcal{M}_k, \mathcal{L}_k)(W_{1k})E_{1k} + \mathfrak{g}_-W_0,$$

so that, as we prove in Proposition 5, we deduce

$$F(\mathcal{M}_k, \mathcal{L}_k)_{>}(E_{11}W) = E_{11}(F(M, L)C_{kk})W + \mathfrak{g}_-W_0 = E_{11}(F(M, L)C_{kk})_+W + \mathfrak{g}_-W_0.$$

Therefore,

$$R := F(\mathcal{M}_k, \mathcal{L}_k)_{>}(E_{11}W) - E_{11}(F(M, L)C_{kk})_+W \in \mathfrak{g}_-W_0$$

and from (8) and (66) we get

$$Rg = F(\mathcal{M}_k, \mathcal{L}_k)_{>}(E_{11}\bar{W}) - E_{11}(F(M, L)C_{kk})_+\bar{W} \in \mathfrak{g}_+\bar{W}_0.$$

so that Proposition 2 implies the first formula in (40). Now, from (70) we get for  $k \neq 1$

$$F(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)_{>}(E_{11}W) = F(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)(W_{1k})E_{1k} + \mathfrak{g}_-W_0.$$

Hence Proposition 5 ensures that

$$R := F(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)_{>}(E_{11}W) - E_{11}(F(M, L)C_{kk})_+W \in \mathfrak{g}_-W_0$$

and from (70) we deduce

$$Rg = F(\bar{\mathcal{M}}_k, \bar{\mathcal{L}}_k)_{>}(E_{11}\bar{W}) - E_{11}(F(M, L)C_{kk})_+\bar{W} \in \mathfrak{g}_+\bar{W}_0.$$

In this way, Proposition 2 leads to the last formula in (40). The proof of (41) follows similarly.  $\square$

## Acknowledgements

The authors wish to thank the Spanish Ministerio de Ciencia e Innovación, research project FIS2008-00200, and acknowledge the support received from the European Science Foundation (ESF) and the activity entitled *Methods of Integrable Systems, Geometry, Applied Mathematics* (MISGAM). MM wish to thank Prof. van Moerbeke and Prof. Dubrovin for their warm hospitality, acknowledge economical support from MISGAM and SISSA and reckons different conversations with P. van Moerbeke, T. Grava, G. Carlet and M. Caffasso. MM also acknowledges to Prof. Liu for his invitation to visit the China Mining and Technology University at Beijing.

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